

MUMBAI UNIVERSITY

SEMESTER – 2

APPLIED MATHEMATICS SOLVED PAPER – DEC 18

N.B:- (1) Question no.1 is compulsory.

(2) Attempt any 3 questions from remaining five questions.

Q.1 a) Evaluate $\int_0^{\infty} \frac{e^{-x^3}}{\sqrt{x}} dx$. [3]

ANS: $I = \int_0^{\infty} \frac{e^{-x^3}}{\sqrt{x}} dx$.

Put $x^3 = t$

$$\therefore x = t^{\frac{1}{3}}$$

$$dx = \frac{1}{3} t^{-\frac{2}{3}}$$

$$\therefore I = \int_0^{\infty} e^{-t} \cdot t^{-\frac{1}{6}} \cdot \frac{1}{3} \cdot t^{-\frac{2}{3}} dt$$

$$\therefore I = \int_0^{\infty} e^{-t} \cdot t^{-\frac{5}{6}} dt$$

$$\therefore I = \frac{1}{3} \left[\frac{1}{6} \right]$$

b) Find the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from $y = 1$ to $y = 2$.

[3]

ANS: We have $x = \frac{y^3}{3} + \frac{1}{4y}$

Diff w.r.t. y , we get

$$\frac{dx}{dy} = y^2 - \frac{1}{4y^2}$$

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$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(y^2 - \frac{1}{4y^2}\right)^2 = y^4 + \frac{1}{2} + \frac{1}{16y^4} = \left(y^2 + \frac{1}{4y^2}\right)^2$$

We know that,

$$s = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$s = \int_1^2 \sqrt{\left(y^2 + \frac{1}{4y^2}\right)^2} dy$$

$$s = \left[\frac{y^3}{3} - \frac{1}{4y}\right]_1^2$$

$$s = \frac{8}{3} - \frac{1}{8} - \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$s = \frac{59}{24}$$

c) Solve $(D^2 + D)y = e^{4x}$. [3]

ANS: For auxiliary equation,

$$D^2 + D = 0$$

Solving we get,

$$D = -1, 0.$$

$$\therefore \text{C.F.} = C_1 e^{-x} + C_2 e^{0x}$$

$$\therefore \text{C.F.} = C_1 e^{-x} + C_2$$

For P.I.,

$$y = \frac{e^{4x}}{D^2 + D}$$

Now, put $D = 4$

$$\therefore y = \frac{e^{4x}}{4^2 + 4} = \frac{e^{4x}}{20}$$

\therefore The complete solution is,

$$y = C_1 e^{-x} + C_2 + \frac{e^{4x}}{20}$$

d) Evaluate $\int_0^1 \int_{x^2}^x xy(x+y)dydx$. [3]

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ANS: We have,

$$I = \int_0^1 \int_{x^2}^x xy(x+y) dy dx.$$

$$I = \int_0^1 \left[\frac{x^2 y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^x dx$$

$$I = \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right] dx$$

$$I = \left[\frac{5}{6} \cdot \frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1$$

$$I = \frac{1}{6} - \frac{1}{14} - \frac{1}{24}$$

$$I = \frac{3}{56}$$

e) Solve $(4x + 3y - 4)dx + (3x - 7y - 3)dy = 0$. [4]

ANS: Given, $(4x + 3y - 4)dx + (3x - 7y - 3)dy = 0$.

$$\therefore M = (4x + 3y - 4) \quad \text{and} \quad N = (3x - 7y - 3)$$

Differentiating M by y and N by x, we get,

$$\frac{dM}{dy} = 3 \quad \text{And} \quad \frac{dN}{dx} = 3$$

$$\therefore \frac{dM}{dy} = \frac{dN}{dx}$$

\therefore The given equations are exact.

For solution,

$$\int M dx = \int (4x + 3y - 4) dx$$

$$\int M dx = 2x^2 + 3xy - 4x$$

$$\int (\text{Term is } N \text{ free from } x) = \int -7y - 3 dy$$

$$= \frac{-7y^2}{2} - 3y$$

\therefore The final solution is,

$$2x^2 + 3xy - 4x - \frac{7y^2}{2} - 3y = c$$

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$$4x^2 + 6xy - 8x - 7y^2 - 6y = c$$

f) Solve $\frac{dy}{dx} = 1 + xy$ with initial condition $x_0 = 0, y_0 = 0.2$ By Taylor's series method. Find the approximate value of y for $x = 0.4$ (step size = 0.4).

ANS: The Taylor series is given by,

$$y = y_0 + xy'_0 + \frac{x^2}{2!} y''_0 + \frac{x^3}{3!} y'''_0 + \dots \quad \dots \quad (1)$$

With $x_0 = 0, y_0 = 0.2, x = 0.4$

$$\text{Now, } y' = 1 + xy \quad \therefore y'_0 = 1$$

$$y'' = y + xy' \quad \therefore y''_0 = y_0 = 0.2$$

$$y''' = y' + y' + xy''$$

$$= 2y' + xy'' \quad \therefore y'''_0 = 2y'_0 = 2$$

$$y'''' = 2y'' + y'' + xy''' \quad \therefore y''''_0 = 3y''_0 + xy'''_0 = 0.6$$

Putting these values in equation 1, we get

$$y = 0.2 + (0.4)1 + \frac{(0.4)^2}{2!} 0.2 + \frac{(0.4)^3}{3!} \cdot 2 + \frac{(0.4)^4}{4!} \cdot (0.6) + \dots$$

$$y = 0.2 + 0.4 + 0.016 + 0.02133 + 0.00064$$

$$y = 0.63797.$$

Q.2 a) Solve $\frac{d^2y}{dx^2} - 16y = x^2 e^{3x} + e^{2x} - \cos 3x + 2^x$. **[6]**

ANS: The auxiliary equation is $D^2 - 16 = 0$

$$\therefore D = 4, -4$$

$$\therefore \text{The C.F. is } y = C_1 e^{4x} + C_2 e^{-4x}$$

Now, to find P.I.,

$$\text{P.I.} = \frac{1}{D^2 - 16} (x^2 e^{3x} + e^{2x} - \cos 3x + 2^x)$$

$$\text{Now, } \frac{1}{D^2 - 16} x^2 e^{3x} = e^{3x} \cdot \frac{1}{(D+3)^2 - 16} \cdot x^2$$

$$= e^{3x} \cdot \frac{1}{D^2 + 6D + 9 - 16} \cdot x^2 = e^{3x} \cdot \frac{1}{D^2 + 6D - 7} \cdot x^2$$

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$$\begin{aligned}
 &= -\frac{e^{3x}}{7} \cdot \frac{1}{\left(1 - \frac{D^2+6D}{7}\right)} \cdot x^2 \\
 &= -\frac{e^{3x}}{7} \cdot \left(1 - \frac{D^2+6D}{7}\right)^{-1} \cdot x^2 \\
 &= -\frac{e^{3x}}{7} \left(1 + \frac{D^2+6D}{7} + \frac{D^4+6D^3+36D^2}{49} + \dots\right) \cdot x^2 \\
 &= -\frac{e^{3x}}{7} \left(x^2 + \frac{12x+2}{7} + \frac{72}{49}\right) = -\frac{e^{3x}}{7} \left(x^2 + \frac{12x}{7} + \frac{86}{49}\right)
 \end{aligned}$$

$$\therefore \frac{1}{D^2-16} \cdot e^{2x} = e^{2x} \frac{1}{2^2-16} = e^{2x} \cdot \frac{1}{2^2-16} = e^{2x} \cdot \frac{1}{12}$$

$$\therefore \frac{1}{D^2-16} \cdot \cos 3x = \frac{\cos 3x}{-9-16} = \frac{\cos 3x}{-25}$$

$$\therefore \frac{1}{D^2-16} \cdot 2^x = \frac{1}{D^2-16} \cdot e^{x \log 2} = \frac{e^{x \log 2}}{(\log 2)^2-16} = \frac{2^x}{(\log 2)^2-16}$$

$$\therefore \text{P.I.} = -\frac{e^{3x}}{7} \left(x^2 + \frac{12x}{7} + \frac{86}{49}\right) + e^{2x} \cdot \frac{1}{12} + \frac{\cos 3x}{25} + \frac{2^x}{(\log 2)^2-16}$$

\therefore The complete equation is,

$$y = C_1 e^{4x} + C_2 e^{-4x} - \frac{e^{3x}}{7} \left(x^2 + \frac{12x}{7} + \frac{86}{49}\right) + e^{2x} \cdot \frac{1}{12} + \frac{\cos 3x}{25} + \frac{2^x}{(\log 2)^2-16}$$

b) Show that $\int_0^{\pi} \frac{\log(1+a \cos x)}{\cos x} dx = \pi \sin^{-1} a$ $0 \leq a \leq 1$. **[6]**

ANS: Let I (a) be the given integral. By the rule of differentiation under the integral sign.

$$\frac{dI}{da} = \int_0^{\pi} \frac{df}{da} dx = \int_0^{\pi} \frac{1}{\cos x} \cdot \frac{\cos x}{1+a \cos x} dx = \int_0^{\pi} \frac{dx}{1+a \cos x}$$

Put $t = \tan \frac{x}{2}$, $dx = \frac{2dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

When $x = 0$, $t = 0$;

When $x = \pi$, $t = \tan \frac{\pi}{2} = \infty$

$$\therefore \frac{dl}{da} = \int_0^{\infty} \frac{1}{1+a \cdot \left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2 dt}{1+t^2}$$

$$\frac{dl}{da} = \int_0^{\infty} \frac{2 dt}{(1+t^2)+a(1-t^2)}$$

$$\frac{dl}{da} = \int_0^{\infty} \frac{2 dt}{(1+a)+(1-a)t^2}$$

$$\frac{dl}{da} = \frac{1}{1-a} \int_0^{\infty} \frac{2 dt}{\left[\frac{1+a}{1-a}\right]+t^2}$$

$$\frac{dl}{da} = \frac{2}{1-a} \sqrt{\frac{1-a}{1+a}} \cdot \left[\tan^{-1} \sqrt{\frac{1-a}{1+a}} \right]_0^{\infty}$$

$$\frac{dl}{da} = \frac{2}{\sqrt{1-a^2}} \cdot \frac{\pi}{2}$$

$$\frac{dl}{da} = \frac{\pi}{\sqrt{1-a^2}}$$

Integrating both sides w.r.t. a, we get

$$I = \pi \sin^{-1} a + c$$

To find c, put a = 0

$$I(0) = \pi \sin^{-1} 0 + c, c = 0$$

$$\therefore I = \pi \sin^{-1} a$$

$$\therefore \int_0^{\pi} \frac{\log(1+a \cos x)}{\cos x} dx = \pi \sin^{-1} a$$

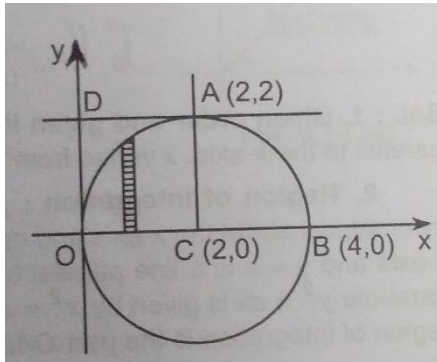
c) Change the order of integration and evaluate $\int_0^2 \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx dy$.
[8]

ANS: 1) Given order and given limits: Given order is: first w.r.t. x and then w.r.t y i.e., a strip parallel to the x-axis varies from $x = 2 - \sqrt{4 - y^2}$ to $x = 2 + \sqrt{4 - y^2}$. Y varies from $y = 0$ to $y = 2$.

2) Region of integration: $x = 2 - \sqrt{4 - y^2}$ is the arc and $x = 2 + \sqrt{4 - y^2}$ is the arc of the circle $(x - 2)^2 + y^2 = 4$ with centre at (2, 0) and radius = 2 above the x-axis. $y = 0$ is the x-axis and $y = 2$ is the line parallel to the x-axis through A (2, 2). The region of integration is the

semi-circle OAB above the x-axis. The points of intersection of the circle and the x-axis are O (0, 0) and B (4, 0).

3) Change of order of integration: To change the order, consider a strip parallel to the y-axis in the region of integration. On this strip y varies from y = 0 to y = $\sqrt{4 - (x - 2)^2}$ and then strip moves from x = 0 to x = 4.



$$I = \int_0^4 \int_0^{\sqrt{4-(x-2)^2}} dy dx$$

$$I = \int_0^4 [y]_0^{\sqrt{4-(x-2)^2}} dx$$

$$I = \int_0^4 \sqrt{4-(x-2)^2} dx$$

$$I = \left[\frac{x-2}{2} \sqrt{4-(x-2)^2} + 2 \sin^{-1} \frac{x-2}{2} \right]_0^4$$

$$I = \left(2 \cdot \frac{\pi}{2} \right) - \left(-2 \cdot \frac{\pi}{2} \right)$$

$\therefore I = 2\pi$

Q.3 a) Evaluate $\iiint (x + y + z) dx dy dz$ over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. [6]

ANS:

$$I = \int_{x=0}^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} (x + y + z) dz dy dx$$

$$I = \int_{x=0}^1 \int_{y=0}^{1-x} \left[\frac{(x+y+z)^2}{2} \right]_0^{1-x-y} dy dz$$

$$I = \frac{1}{2} \int_{x=0}^1 \int_{y=0}^{1-x} [1 - (x + y)^2] dy dx$$

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$$I = \frac{1}{2} \int_{x=0}^1 \left[y - \frac{(x+y)^2}{2} \right]_0^{1-x} dx$$

$$I = \frac{1}{2} \int_{x=0}^1 \left[(1-x) - \frac{1}{3} + \frac{x^3}{3} \right] dx$$

$$I = \frac{1}{2} \left[\frac{2x}{3} - \frac{x^2}{2} + \frac{x^4}{12} \right]_0^1$$

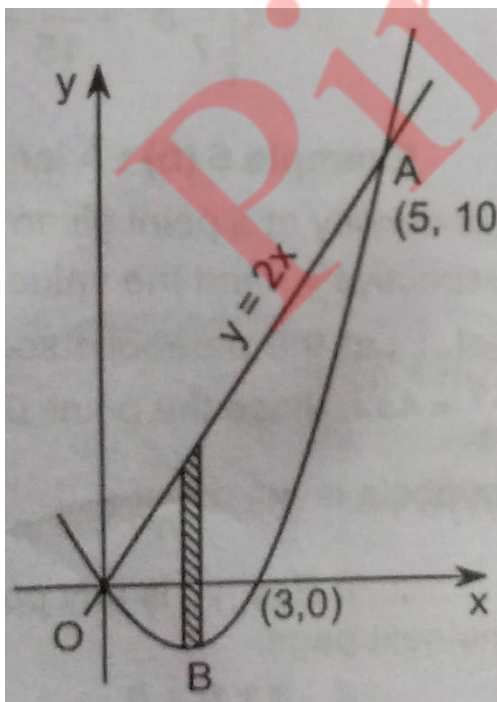
$$I = \frac{1}{2} \cdot \frac{3}{12} = \frac{1}{8}$$

$\therefore I = \frac{1}{8}$

b) Find the mass of lamina bounded by the curves $y = x^2 - 3x$ and $y = 2x$ if the density of the lamina at any point is given by $\frac{24}{25}xy$.

[6]

ANS: The curve $y = x^2 - 3x$ i.e. $y + \frac{9}{4} = (x - \frac{3}{2})^2$ is parabola intersecting the x-axis in $x = 0$ and $x = 3$. The line $y = 2x$ intersects this parabola at $x^2 - 3x = 2x$ i.e. $x^2 - 5x = 0$ i.e. at $x = 0, x = 5$. Therefore, points of intersection are $(0,0)$ and $(5,10)$. The surface density is $\rho = (24/25)xy$. Taking the elementary strip parallel to the y-axis, on the strip y varies from $y = x^2 - 3x$ to $y = 2x$ and then x varies from $x = 0$ to $x = 5$.



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$$\begin{aligned}
 \therefore \text{Mass of lamina} &= \int_0^5 \int_{x^2-3x}^{2x} \frac{24}{25} xy dx dy \\
 &= \frac{24}{25} \int_0^5 x \left[\frac{y^2}{2} \right]_{x^2-3x}^{2x} dx \\
 &= \frac{24}{50} \int_0^5 4x^3 - x(x^4 - 6x^3 + 9x^2) dx \\
 &= \frac{24}{50} \int_0^5 -5x^3 + 6x^4 - x^5 dx \\
 &= \frac{24}{50} \left[-\frac{x^4}{4} + \frac{6x^5}{5} - \frac{x^6}{6} \right]_0^5 \\
 &= \frac{24}{50} \cdot 5^4 \left[-\frac{25}{6} + 6 - \frac{5}{4} \right] \\
 &= \frac{24}{50} \cdot 5^4 \cdot \frac{7}{12}
 \end{aligned}$$

\therefore Mass of lamina = 175.

c) Solve $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + 3y = \frac{\log x \cdot \cos(\log x)}{x}$ **[8]**

ANS: Given that,

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + 3y = \frac{\log x \cdot \cos(\log x)}{x}$$

Putting $z = \log x$ and $x = e^z$, we get

$$[D(D-1) + 3D + 3]y = e^{-z} \cdot z \cdot \cos z$$

$$[D^2 + 2D + 3]y = e^{-z} \cdot z \cdot \cos z$$

$$\therefore \text{The A.E. is } D^2 + 2D + 3 = 0$$

$$\therefore D = \frac{-2 \pm 2\sqrt{2}i}{2} = -1 \pm \sqrt{2}i$$

$$\therefore \text{The C.F. is } y = e^{-z} (C_1 \cos \sqrt{2}z + C_2 \sin \sqrt{2}z)$$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 3} e^{-z} \cdot z \cdot \cos z$$

$$= e^{-z} \cdot \frac{1}{(D-1)^2 + 2(D-1) + 3} \cdot z \cdot \cos z = e^{-z} \cdot \frac{1}{D^2 + 2} \cdot z \cdot \cos z$$

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$$\begin{aligned}
 &= e^{-z} \left[z - \frac{1}{D^2+2} \cdot 2D \right] \cdot \frac{1}{D^2+2} \cdot \cos z \\
 &= e^{-z} \left[z - \frac{1}{D^2+2} \cdot 2D \right] \cos z = e^{-z} \left[z \cos z + \frac{1}{D^2+2} \cdot 2 \sin z \right] \\
 &= e^{-z} [z \cos z + 2 \sin z]
 \end{aligned}$$

The complete solution is,

$$y = \text{C.F.} + \text{P.I.}$$

$$y = e^{-z} (C_1 \cos \sqrt{2}z + C_2 \sin \sqrt{2}z) + e^{-z} [z \cos z + 2 \sin z]$$

$$\begin{aligned}
 y = &\frac{1}{x} (C_1 \cos \sqrt{2} \log x + C_2 \sin \sqrt{2} \log x) + \frac{1}{x} [\log x \cos \log x \\
 &+ 2 \sin \log x]
 \end{aligned}$$

Q.4 a) Find by double integration the area bounded by the parabola

$$y^2 = 4x \text{ And } y = 2x - 4 \quad [6]$$

ANS: The parabola $y^2 = 4x$ and the line $y = 2x - 4$ intersect where
 $(2x - 4)^2 = 4x$

$$\therefore 4x^2 - 16x + 16 = 4x \quad \therefore 4x^2 - 20x + 16 = 0$$

$$\therefore x^2 - 5x + 4 = 0 \quad \therefore (x - 4)(x - 1) = 0$$

$$\therefore x = 1, 4.$$

When $x = 1$, $y = 2 - 4 = -2$; and when $x = 4$, $y = 8 - 4 = 4$. Thus, the points of intersection are A (1, -2) and B (4, 4).

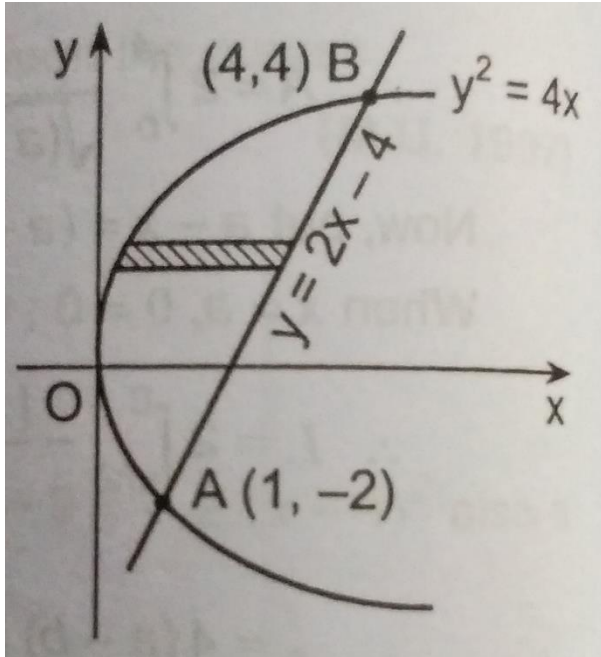
Now, consider a strip parallel to x-axis. On this strip x varies from $x = y^2/4$ to $x = (y+4)/2$. The strip then moves parallel to the x-axis from $y = -2$ to $y = 4$.

$$\begin{aligned}
 \therefore A &= \int_{-2}^4 \int_{y^2/4}^{(y+4)/2} dx dy = \int_{-2}^4 \left[x \right]_{\frac{y^2}{4}}^{\frac{y+4}{2}} dy \\
 &= \int_{-2}^4 \left(\frac{y+4}{2} - \frac{y^2}{4} \right) dy \\
 &= \frac{1}{4} \int_{-2}^4 (2y + 8 - y^2) dy
 \end{aligned}$$

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$$\begin{aligned}
 &= \frac{1}{4} \left[y^2 + 8y - \frac{y^3}{3} \right]_{-2}^4 \\
 &= \frac{1}{4} \left[\left(16 + 32 - \frac{64}{3} \right) - \left(4 - 16 + \frac{8}{3} \right) \right] \\
 &= \frac{1}{4} (60 - 24)
 \end{aligned}$$

$\therefore A = 9$

b) Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$ [6]

ANS: Given, $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Dividing both sides by $\cos^2 y$,

$$\sec^2 y \frac{dy}{dx} + x \sec^2 y \sin 2y = x^3$$

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \dots\dots\dots (1)$$

Put $\tan y = v$ and differentiate w.r.t. x ,

$$\sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$$

Hence, from (1), we get $\frac{dv}{dx} + 2v \cdot x = x^3$

$$\therefore P = 2x \text{ And } Q = x^3$$

$$\therefore \int P dx = \int 2x dx = x^2$$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\int 2x dx} = e^{x^2}$$

$$\therefore \text{The solution is } v e^{x^2} = \int e^{x^2} x^3 dx + c$$

To find the integral put $x^2 = t$, $xdx = \frac{dt}{2}$.

$$\therefore I = \int e^t \cdot t \cdot \frac{dt}{2} = \frac{1}{2} [te^t - \int e^t \cdot dt] \dots \dots \dots \text{ [By parts]}$$

$$\therefore I = \frac{1}{2} [te^t - e^t] = \frac{1}{2} e^t (t - 1) = \frac{1}{2} e^{x^2} (x^2 - 1)$$

$$\therefore \text{The solution is } v e^{x^2} = \frac{1}{2} e^{x^2} (x^2 - 1) + c$$

$$\therefore \tan y e^{x^2} = \frac{1}{2} e^{x^2} (x^2 - 1) + c$$

c) Solve $\frac{dy}{dx} = x^3 + y$ with initial conditions $y(0) = 2$ at $x = 0.2$ in step of

$h = 0.1$ by Runge Kutta method of Fourth order. [8]

ANS: Given that, $\frac{dy}{dx} = x^3 + y$

$$f(x, y) = x^3 + y, x_0 = 0, y_0 = 2 \text{ and } h = 0.1$$

$$\therefore k_1 = hf(x_0, y_0) = 0.1(0 + 2) = 0.2$$

$$\therefore k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\left[\left(\frac{0.1}{2}\right)^3 + 2 + \frac{0.2}{2}\right] = 0.2100$$

$$\therefore k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1\left[\left(\frac{0.1}{2}\right)^3 + 2 + \frac{0.2100}{2}\right] = 0.2105$$

$$\therefore k_4 = hf\left(x_0 + h, y_0 + k_3\right) = 0.1\left[\left(\frac{0.1}{2}\right)^3 + 2 + \frac{0.2100}{2}\right] = 0.23105$$

$$\therefore k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = \frac{0.2 + 2(0.21) + 2(0.2105) + 0.23105}{6}$$

$$\therefore k = 0.2120$$

Q.5 a) Evaluate $\int_0^1 x^5 \sin^{-1} x dx$ and find the value of $\beta\left(\frac{9}{2}, \frac{1}{2}\right)$. [6]

ANS: $I = \int_0^1 x^5 \sin^{-1} x dx$

Put $\sin^{-1} x = t \quad \therefore x = \sin t \quad dx = \cos t dt$

When $x = 0, t = 0$ when $x = 1, t = \pi/2$

$$I = \int_0^{\pi/2} \sin^5 t \cdot t \cdot \cos t dt = \int_0^{\pi/2} t(\sin^5 t \cdot \cos t) dt$$

Integrating by parts,

$$I = \left[t \cdot \frac{\sin^6 x}{6} \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin^6 x}{6} \cdot 1 \cdot dt$$

$$I = \left(\frac{\pi}{2} \cdot \frac{1}{6} - 0 \right) - \frac{1}{6} \cdot \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$I = \frac{\pi}{12} - \frac{5\pi}{192}$$

$\therefore I = \frac{11\pi}{192}$

$$\beta\left(\frac{9}{2}, \frac{1}{2}\right) = \frac{\sqrt{\frac{9}{2}} \sqrt{\frac{1}{2}}}{\sqrt{5}} = \frac{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2}}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$\beta\left(\frac{9}{2}, \frac{1}{2}\right) = \frac{105\pi}{384}$

b) In a circuit containing inductance L, resistance R, and voltage E, the current i is given by $L \frac{di}{dt} + Ri = E$. Find the current i at time t at t = 0 and i = 0 and L, R and E are constants. [6]

ANS: The given equation $\frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$ is linear of the type $\frac{dy}{dx} + Py = Q$

\therefore Its solution is $ie^{\int R/L dt} = \int e^{\int R/L dt} \cdot \frac{E}{L} \cdot dt + c$

$$i \cdot e^{Rt/L} = \frac{E}{L} \int e^{Rt/L} dt + c = \frac{E}{L} \cdot e^{Rt/L} \frac{L}{R} + c$$

$$= \frac{E}{R} e^{Rt/L} + c$$

When $t = 0$ and $i = 0 \therefore c = -\frac{E}{R}$

$$\therefore i \cdot e^{Rt/L} = \frac{E}{R} e^{Rt/L} - \frac{E}{R}$$

$$\therefore i = \frac{E}{R} (e^{Rt/L} - 1)$$

$$\therefore i = \frac{E}{R} (1 - e^{-Rt/L})$$

c) Evaluate $\int_0^6 \frac{dx}{1+3x}$ by using 1) Trapezoidal 2) Simpsons (1/3) rd. and 3) Simpsons (3/8) Th rule. [8]

ANS:

X	0	1	2	3	4	5	6
Y	1	0.25	0.1428	0.1	0.0769	0.0625	0.0526
Ordinate	y_0	y_1	y_2	y_3	y_4	y_5	y_6

1) Trapezoidal Rule:

$$I = \frac{h}{2} (X + 2R)$$

$X = \text{Sum of extreme value} = 1 + 0.0526 = 1.0526$

$R = \text{Sum of Remaining values} = 0.25 + 0.1428 + 0.1 + 0.0769 + 0.0625 = 0.6322$

$$I = \frac{1}{2} (1.0526 + 2(0.6322))$$

$$I = 1.1585$$

2) Simpsons (1/3) rd rule

$$I = \frac{h}{3} (X + 2E + 4O)$$

$X = \text{Sum of Extreme values} = 1 + 0.0526 = 1.0526$

$E = \text{Sum of even ordinates} = 0.1428 + 0.0769 = 0.2197$

$O = \text{Sum of odd ordinates} = 0.25 + 0.1 + 0.0625 = 0.4125$

$$I = \frac{1}{3} (1.0526 + 2(0.2197) + 4(0.4125))$$

$$I = 0.5616.$$

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3} Simpsons (3/8) Th rule.

$$I = \frac{3h}{8} (X + 2T + 4R)$$

X = Sum of extreme value = 1 + 0.0526 = 1.0526

T = Sum of multiple of three = 0.1

R = Sum of Remaining values = 0.25 + 0.1428 + 0.0769 + 0.0625 = 0.5322

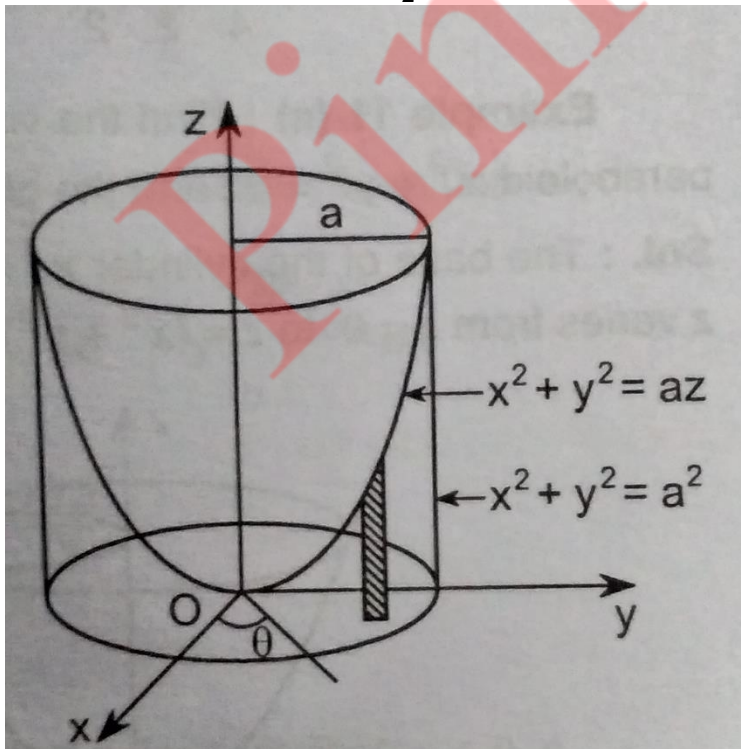
$$I = \frac{3 \times 1}{8} (1.0526 + 2(0.1) + 4(0.5322))$$

$$I = 1.06845.$$

Q.6 a) Find the volume bounded by the paraboloid $x^2 + y^2 = az$ and the cylinder $x^2 + y^2 = a^2$. [6]

ANS: The equations of the cylinder and the paraboloid in polar form are $r = a$ and $r^2 = az$.

Now, z varies from $z = 0$ to $z = r^2/a$, r varies from $r = 0$ to $r = a$ and θ varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$ taken 4 times.



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$$\therefore V = 4 \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^a \int_{z=0}^{r^2/a} r \, dr \, d\theta \, dz$$

$$\therefore V = 4 \int_{\theta=0}^{\frac{\pi}{2}} r [z]_0^{r^2/a} \, dr \, d\theta$$

$$\therefore V = 4 \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^a \frac{r^3}{a} \, dr \, d\theta$$

$$\therefore V = \frac{4}{a} \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{r^4}{4} \right]_0^a \, d\theta$$

$$\therefore V = \frac{4}{a} \int_0^{\frac{\pi}{2}} \frac{a^4}{4} \, d\theta$$

$$\therefore V = a^3 \int_0^{\frac{\pi}{2}} d\theta$$

$$\therefore V = a^3 [\theta]_0^{\pi/2}$$

$$\therefore V = \frac{\pi a^3}{2}$$

b) Change to polar coordinates and evaluate $\int_0^1 \int_0^x (x+y) \, dy \, dx$.

[6]

ANS: 1) Region of integration: $y=0$ is the x-axis and $y=x$ is a line OB through the origin; $x=0$ is the y-axis and $x=1$ is a line AB parallel to the y-axis. Thus the region of integration is the triangle OAB.

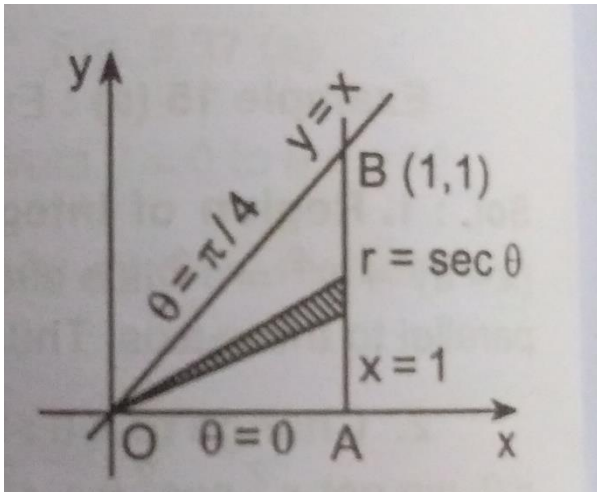
2) Change to r, θ : Putting $x = r \cos \theta$ and $y = r \sin \theta$, the line $y = x$ becomes $r \sin \theta = r \cos \theta$ i.e. $\tan \theta = 1$ i.e. $\theta = \frac{\pi}{4}$. The x-axis is given by $\theta = 0$ and the y-axis is given by $\theta = \frac{\pi}{2}$. And line $x = 1$ is given by $r \cos \theta = 1$ i.e. $r = \sec \theta$.

3) Integrand: Putting $x = r \cos \theta$ and $y = r \sin \theta$ in $(x+y)$, we get, $r \cos \theta + r \sin \theta = r(\cos \theta + \sin \theta)$ and $dy \, dx$ is replaced by $r \, dr \, d\theta$

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$$\therefore I = \int_0^{\frac{\pi}{4}} \int_0^{\sec \theta} r(\cos \theta + \sin \theta) r dr d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \int_0^{\sec \theta} (\cos \theta + \sin \theta) r^2 dr d\theta$$

$$I = \int_0^{\frac{\pi}{4}} (\cos \theta + \sin \theta) \left[\frac{r^3}{3} \right]_0^{\sec \theta} d\theta$$

$$I = \frac{1}{3} \int_0^{\frac{\pi}{4}} (\cos \theta + \sin \theta) \sec^3 \theta d\theta$$

$$I = \frac{1}{3} \left[\int_0^{\frac{\pi}{4}} \sec^2 \theta d\theta + \int_0^{\frac{\pi}{4}} \frac{1}{\cos^3 \theta} \sin \theta d\theta \right]$$

$$I = \frac{1}{3} \left[\tan \theta + \frac{1}{2\cos^2 \theta} \right]_0^{\frac{\pi}{4}}$$

$$I = \frac{1}{3} \left(1 + \frac{1}{2}(2-1) \right)$$

$$I = \frac{1}{2}$$

c) Solve by method of variation of parameters

$$\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 2y = e^{e^x} \quad [8]$$

ANS: A.E: $D^2 + 3D + 2 = 0$

Solving the equation, we get

$$\therefore D = -1, -2.$$

$$\therefore \text{C.F} = C_1 e^{-x} + C_2 e^{-2x}.$$

$$\therefore y_1 = e^{-x} \quad y_2 = e^{-2x}$$

$$\therefore y_1' = -e^{-x} \quad y_2' = -2e^{-2x}$$

$$\begin{aligned} \therefore W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} \\ &= -2e^{-3x} + e^{-3x} \\ &= -e^{-3x} \end{aligned}$$

$$X = e^{e^x}.$$

$$\begin{aligned} \therefore u &= -\int \frac{y_2 X}{W} dx \\ &= -\int \frac{e^{-2x} e^{e^x}}{-e^{-3x}} dx \\ &= -\int e^{e^x} \cdot e^x dx \end{aligned}$$

$$\text{Put } e^x = t$$

$$e^x dx = dt$$

$$\therefore \int e^t dt = e^t + c.$$

$$\therefore W = e^{e^x} + c$$

$$v = \int \frac{y_1 X}{W} dx$$

$$v = \int \frac{e^{-x} e^{e^x}}{-e^{-3x}} dx$$

$$v = \int e^{e^x} e^{2x} dx$$

$$\text{Putting } e^x = t$$

$$\therefore v = \int e^t \cdot t dt = te^t - e^t$$

$$\therefore v = e^x e^{e^x} - e^{e^x}$$

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$$\begin{aligned}\therefore \text{P.I.} &= uy_1 + vy_2 = e^{e^x} \cdot e^{-x} - (e^x e^{e^x} - e^{e^x}) e^{-2x} \\ &= e^{-2x} \cdot e^{e^x}\end{aligned}$$

\therefore The complete solution is,

$$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1 e^{-x} + C_2 e^{-2x} + e^{-2x} \cdot e^{e^x}$$

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